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# Charge–monopole trajectories and the WKB approximation

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**Abstract**

The classical nonrelativistic motion of an electric charge in the presence of a magnetic monopole is reviewed. Using general properties of these trajectories it is shown that the path integral form of the WKB approximation for the charged particle propagator develops an anomalous form for the transition probability density unless the Dirac quantization condition is imposed. This result is shown to be independent of the specific form of the vector potential used to represent the magnetic monopole. The general result is then verified for the specific case that the standard Dirac string solution is used in the path integral. This result is interpreted as an indirect demonstration of the Dirac condition. Other properties of the solutions and the associated action are briefly discussed.

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## 1. Introduction

Since the seminal work of Dirac [1] the quantum mechanical aspects of magnetic monopoles have been studied in a variety of contexts. The Dirac quantization condition for electric charge $e$ and magnetic charge $g$,

$$eg = \tfrac{1}{2}n, \tag{1}$$

where $n$ is an integer, has been subsequently derived by a variety of methods [2–4] and generalized to nonabelian theories [5]. Dirac's original argument for (1) was based on a singular form for the vector potential $\mathbf{A}$ in the presence of the magnetic monopole—the 'Dirac string.' Rotational invariance of the electron wavefunction in the presence of such a vector potential then requires (1).

The nature of the Dirac condition remains a subject of investigation. There has been recent controversy regarding the nature of the singularities required by the vector potential [6]. This issue is important since many derivations of the Dirac condition, such as Wu and Yang's fibre bundle formulation [4], rely on patching a specific form for the vector potential to avoid its singularities. Alternate derivations [3] of the Dirac condition and its generalizations stem from enforcing the rotation group algebra—either using finite-dimensional or infinite-dimensional representations—on the generalization of the total quantum mechanical angular momentum $\mathbf{J} = \mathbf{r} \times (-\mathrm{i}\nabla - e\mathbf{A}) - eg\mathbf{r}/r$, which also directly involves the specific form of the vector potential $\mathbf{A}$. As a result, any derivation of (1) that does not require a specific form for the vector potential allows the questions regarding the role and nature of the singularities of $\mathbf{A}$ to be sidestepped. One such demonstration has been given by Jackiw [7], who examined the cocycles of the electron wavefunction that arise from demanding associativity of translations.

This paper will examine the relationship of the path integral form of the WKB approximation to condition (1) in a manner that is also ultimately independent of the specific form for the vector potential. The relationship is developed using the nature of classical trajectories for a spinless electric charge moving in the presence of a magnetic monopole. Some aspects of large impact parameter trajectories are discussed at the textbook level [8], but to the knowledge of the author there has been no systematic consideration of the quantum mechanical implications of classical trajectories in the charge–monopole system. It will be demonstrated that for the case that the initial and final positions are parallel there are multiple classical trajectories characterized by a winding number. A general proof then shows that the path integral form of the WKB approximation for the electric charge's propagator yields an anomalous transition probability unless the Dirac condition is met. This proof is independent of a specific form for the vector potential, although it will be shown to be consistent with the use of the Dirac string vector potential in the path integral. This result is interpreted as indirect support for the necessity of the Dirac condition (1), since it is required to salvage the WKB approximation. It has the added aspect of being independent of the specific form of the vector potential.

The remainder of the paper is organized as follows. In section 2 relevant aspects of the classical trajectories are reviewed. In section 3 general properties of the path integral representation of the WKB approximation and the classical trajectories are used to show that the electron propagator yields a unique transition probability only if the Dirac condition is met. Other aspects of the propagator are briefly discussed.

## 2. Classical trajectories

The classical motion of a spinless charge in the presence of magnetic monopole fixed at the origin—or equivalently the reduced mass problem—was first analysed by Poincarè [9] and subsequently considered by other authors [10]. The properties of trajectories between an initial position $\mathbf{r}_o$ and a final position $\mathbf{r}_f$ in the time $T$ will be reviewed briefly so that results relevant to this paper can be developed.

For a magnetic monopole with charge $g$ fixed at the origin, so that the magnetic field is $\mathbf{B} = g\mathbf{r}/r^3$, a spinless electric point charge $e$ with mass $m$ has the classical equation of motion

$$\dot{\mathbf{p}} = m\ddot{\mathbf{r}} = e\dot{\mathbf{r}} \times \mathbf{B} = \frac{eg}{r^3}\dot{\mathbf{r}} \times \mathbf{r} = -\frac{eg\mathbf{L}}{mr^3}, \tag{2}$$

where $\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}}$ is the usual mechanical angular momentum. For simplicity it will be assumed that $eg > 0$, although the opposite sign is possible. Since $\mathbf{p}\cdot\mathbf{L} = 0$ the kinetic energy $E = p^2/2m$ is conserved. Since $\mathbf{r}\cdot\mathbf{L} = 0$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\cdot\mathbf{p}) = \frac{p^2}{m} = 2E \quad \Longrightarrow \quad \mathbf{r}\cdot\mathbf{p} = R + 2Et, \tag{3}$$

where $R$ is the initial value of $\mathbf{r}\cdot\mathbf{p}$. Integrating (3) yields $r$ as a function of time

$$r(t) = \sqrt{r_o^2 + \frac{2Rt}{m} + \frac{2Et^2}{m}}. \tag{4}$$

If $v_o$ is the initial speed of the electric charge then the radial velocity $v_r \to v_o t$ for large $t$.

Using (2) it is straightforward to show that the vector

$$\mathbf{D} = \mathbf{L} - eg\hat{\mathbf{r}} \tag{5}$$

is a constant of motion, where $\hat{\mathbf{r}} = \mathbf{r}/r$ is a radial unit vector pointing from the origin to the current position of the electric charge. Unlike the central force problem, there are no stable planar orbits since $\mathbf{L}$ is not conserved. However, because $D^2 = L^2 + e^2g^2$ is constant it follows that $L^2$ is constant. The angle $\alpha$ between $\mathbf{L}$ and $\mathbf{D}$ is constant since $\mathbf{L}\cdot\mathbf{D} = LD\cos\alpha = L^2$, so that

$$\cos\alpha = \frac{L}{D} = \frac{L}{\sqrt{L^2 + e^2g^2}}. \tag{6}$$

Since $\hat{\mathbf{r}}\cdot\mathbf{D} = -eg$ is a constant, it follows that the angle between $\hat{\mathbf{r}}$ and $\mathbf{D}$ is also constant, demonstrating that the classical trajectory lies on a cone with its tip at the origin and its axis coinciding with $-\mathbf{D}$. The half-angle of the cone, denoted by $\psi$, is then determined from $L$ by $\cos\psi = eg/D = eg/\sqrt{L^2 + e^2g^2}$. Because $\mathbf{r}$ lies in the cone and the velocity is in the tangent space of the cone it follows that $\mathbf{L}$ is perpendicular to the surface of the cone at all points along the trajectory. This yields $\cos\alpha = \sin\psi$ and, when combined with result (6), gives $L = eg\tan\psi$. Using the relation of $\mathbf{L}$ and $\mathbf{D}$ shows that for the case $eg > 0$ the motion along the cone is right handed around the $\mathbf{D}$ axis.

For the purposes of this paper there is no loss in generality from choosing the $z$-axis to coincide with $-\mathbf{D}$, placing the cone of motion in the $+z$ half-space. For such a choice of coordinates the azimuthal speed is given by $v_\varphi = L/mr$, so that the angular velocity of rotation is left handed about the $z$-axis and given by

$$\dot{\varphi} = -\frac{L}{mr^2\sin\psi}. \tag{7}$$

Using result (4) and the identity

$$L^2 + R^2 = 2mr_o^2E \tag{8}$$

allows (7) to be integrated to obtain the change in the azimuthal angle of the trajectory

$$\varphi(T) - \varphi(0) = -\frac{1}{\sin\psi}\left[\arctan\left(\frac{R + 2ET}{L}\right) - \arctan\left(\frac{R}{L}\right)\right]. \tag{9}$$

The next step in determining the classical trajectory is to relate the values of $E$, $R$ and $\psi$ to the values of $\mathbf{r}_o$, $\mathbf{r}_f$ and $T$. The kinetic energy $E$ is fixed by solving (4) for $v_o^2$ at the time $T$ using the identity (8), with the result that

$$\frac{2E}{m} = v_o^2 = \frac{r_f^2 + r_o^2}{T^2} \pm \frac{2r_f r_o}{T^2}\sqrt{1 - z^2\tan^2\psi}, \tag{10}$$

where the dimensionless parameter $z = egT/mr_o r_f$ has been introduced. The choice of sign in (10) is dictated by the angle between $\mathbf{r}_o$ and $\mathbf{r}_f$, which will be denoted $\gamma$. If $\gamma < \pi/2$ the negative sign is chosen. Although $\psi$ depends upon $z$, it is clear from the conical nature of the motion that the minimum real value for $\psi$ is $\gamma/2$. As a result, the kinetic energy becomes complex if $z\tan(\gamma/2) > 1$, showing that not all initial and final points are connected by a classical trajectory for an arbitrary value of $T$. This is discussed further at the end of section 3.

Result (9) may be simplified by using (8) and (10) and straightforward trigonometric manipulations to show that

$$\varphi(T) - \varphi(0) = -\frac{1}{\sin\psi} \arcsin(z \tan\psi). \tag{11}$$

Result (11) is consistent with the previously determined emergence of complex values for the case that $z\tan(\gamma/2) > 1$. For the choice of coordinate system the value of (11) must coincide with the difference in azimuthal angles for $\mathbf{r}_o$ and $\mathbf{r}_f$ modulo $2\pi$. Since $\psi$ is the constant polar angle of both vectors, its value can therefore be found by solving the spherical coordinate formula

$$\cos\gamma = \cos^2\psi + \sin^2\psi \cos\left(\frac{1}{\sin\psi} \arcsin(z\tan\psi)\right) \tag{12}$$

Solving (12) for arbitrary $z$ and $\gamma$ is not possible and numerical methods must be employed. However, for the case that $\mathbf{r}_o$ and $\mathbf{r}_f$ are parallel, i.e., $\gamma \to 0$, it is relatively simple to demonstrate the existence of multiple trajectories. For the case $\gamma = 0$ there is a radial trajectory given by

$$r = \left(\frac{r_f - r_o}{T}\right) t + r_o \tag{13}$$

corresponding to linear motion. In addition, there is a second trajectory if

$$\varphi(T) - \varphi(0) = -\frac{1}{\sin\psi} \arcsin(z\tan\psi) = -2\pi k, \tag{14}$$

where $k$ is an arbitrary positive integer, since for such a case the right-hand side of (12) reduces to unity, corresponding to $\gamma = 0$. If $\psi \to 0$ so that $z\tan\psi \to 0$, then condition (14) reduces to

$$\cos\psi = \frac{z}{2\pi k} \Longrightarrow z\tan\psi = \sqrt{4\pi^2 k^2 - z^2}. \tag{15}$$

Result (15) shows that for $z \approx 2\pi k$ there is a cone of motion meeting the criterion $\psi \approx 0$ which corresponds to a trajectory that winds around the infinitesimal cone $k$ times. This result occurs since $\gamma = 0$ corresponds to a cumulation point of the motion. Numerical analysis verifies that there are multiple trajectories for $z$ in a range of values around $2\pi k$, but that in the general case these trajectories are energetically distinct from (13) since $\psi$, although small, is nonzero. This is similar to the motion of a free particle on a circle, where for a given time $T$ there is a denumerable infinity of energetically distinct classical trajectories between any two points on the circle characterized by an integer winding number. The existence of multiple trajectories lies at the source of the WKB anomaly discussed in section 3.

## 3. The WKB approximation and the Dirac condition

For the case under consideration, the classical action is given by [11]

$$I = \int_0^T \mathrm{d}t\mathcal{L} = \int_0^T \mathrm{d}t \left(\frac{1}{2} m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + e\mathbf{A}\cdot\dot{\mathbf{r}}\right), \tag{16}$$

where $\mathbf{A}$ is the vector potential of the monopole. The general result derived in this section will not rely upon a specific form for $\mathbf{A}$. Instead, the relationship $\mathbf{B} = \nabla\times\mathbf{A} = g\mathbf{r}/r^3$, so that $\nabla\cdot\mathbf{B} = 4\pi g\delta(\mathbf{x})$, and the time independence of $\mathbf{A}$ will be sufficient.

The charged particle's propagator is given by the Lagrangian path integral [12]

$$\langle \mathbf{r}_f, T|\mathbf{r}_o, 0\rangle = \int_{\mathbf{r}_o}^{\mathbf{r}_f} [\mathrm{d}^3 r] \exp(\mathrm{i}I), \tag{17}$$

where $I$ is given by (16). The WKB approximation for the propagator (17) is given by [13]

$$\langle \mathbf{r}_f, T|\mathbf{r}_o, 0\rangle = \sum_n \left( \frac{\mathrm{i}}{8\pi^3} \det \frac{\partial^2 I_n\left[\mathbf{r}_f, \mathbf{r}_o, T\right]}{\partial x_f^j \partial x_o^k} \right)^{1/2} \exp(\mathrm{i} I_n), \tag{18}$$

where the sum is over all possible classical trajectories connecting $\mathbf{r}_f$ and $\mathbf{r}_o$ in the time $T$ while $I_n[\mathbf{r}_f, \mathbf{r}_o, T]$ is the value of the action along the $n$th trajectory. Because the kinetic energy is constant and $\mathbf{A}$ is time-independent, the action (16) along a classical trajectory becomes

$$I = ET + e\int_P \mathbf{A}\cdot \mathrm{d}\mathbf{r} \tag{19}$$

where $E$ is the kinetic energy, given by (10), and the integral is evaluated along the classical trajectory $P$ joining the initial and final points.

In the previous section it was specifically established that there can be two trajectories, denoted by $P_1$ and $P_2$, for the case that $\mathbf{r}_o$ and $\mathbf{r}_f$ are parallel. For the case $\gamma = 0$ the determinants in (18) are given explicitly by

$$\lim_{\gamma\to 0} \det \frac{\partial^2 I_n\left[\mathbf{r}_f, \mathbf{r}_o, T\right]}{\partial x_f^j \partial x_o^k} = \frac{m^3 z^4}{16 T^3 \sin^4 \frac{1}{2} z}. \tag{20}$$

It is interesting to note that (20) is singular at precisely the values of $z$ that correspond to the energetically degenerate trajectories discussed in the section II. However, the results presented next will not rely upon result (20). Instead, it will be assumed only that there are two trajectories between the initial and final points and that the determinants, denoted by $D_n$ for the respective paths, are real-valued functions. For such a case (18) can then be written as

$$\langle \mathbf{r}_f, T|\mathbf{r}_o, 0\rangle = \sqrt{\frac{\mathrm{i}D_1}{8\pi^3}} \exp\left(\mathrm{i}E_1 T + \mathrm{i}e\int_{P_1} \mathbf{A}\cdot \mathrm{d}\mathbf{r}\right) + \sqrt{\frac{\mathrm{i}D_2}{8\pi^3}} \exp\left(\mathrm{i}E_2 T + \mathrm{i}e\int_{P_2} \mathbf{A}\cdot \mathrm{d}\mathbf{r}\right), \tag{21}$$

where $E_n$ is the kinetic energy of the respective path. However, it is the transition probability density that must be unique, and it is given by

$$|\langle \mathbf{r}_f, T|\mathbf{r}_o, 0\rangle|^2 = \frac{1}{8\pi^3}\left( D_1 + D_2 + 2\sqrt{D_1 D_2} \cos\left( (E_1 - E_2)T + e\oint_P \mathbf{A}\cdot \mathrm{d}\mathbf{r}\right)\right) \tag{22}$$

where the closed path $P$ is defined by the two distinct paths $P_1$ and $P_2$ from $\mathbf{r}_o$ to $\mathbf{r}_f$, i.e.,

$$\oint_P \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \int_{P_1} \mathbf{A}\cdot \mathrm{d}\mathbf{r} - \int_{P_2} \mathbf{A}\cdot \mathrm{d}\mathbf{r}. \tag{23}$$

Using (23) shows that (22) possesses the required invariance under the exchange $1 \leftrightarrow 2$. Stokes' theorem and $\mathbf{B} = \nabla \times \mathbf{A}$ immediately reduces (22) to

$$|\langle \mathbf{r}_f, T|\mathbf{r}_o, 0\rangle|^2 = \frac{1}{8\pi^3}\left\{ D_1 + D_2 + 2\sqrt{D_1 D_2} \cos\left( (E_1 - E_2)T + e\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S}\right)\right\}, \tag{24}$$

where $S$ is an *arbitrary* surface bounded by $P$. Result (24) is true regardless of the specific form for $\mathbf{A}$.

The right handed surfaces bounded by the oriented closed path $P$ fall into two equivalence classes in the presence of a magnetic monopole: those with positive magnetic flux and those with negative magnetic flux. Stated another way, two surfaces bounded by $P$ belong to the same equivalence class if the difference in their surface integrals results in a closed surface integral whose volume does not contain the monopole; otherwise they are in separate equivalence classes. In order for (24) to be unambiguous the choice of equivalence class for the surface integral must be irrelevant. If $S_1$ belongs to one equivalence class and $S_2$ to the other, the absence of an anomaly for (24) requires that

$$\cos\left((E_1 - E_2)T + e\int_{S_1} \mathbf{B}\cdot \mathrm{d}\mathbf{S}\right) - \cos\left((E_1 - E_2)T + e\int_{S_2} \mathbf{B}\cdot \mathrm{d}\mathbf{S}\right) = 0. \tag{25}$$

The identity $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$ shows that (25) is satisfied if

$$\sin\left(\frac{1}{2}e\int_{S_1} \mathbf{B}\cdot \mathrm{d}\mathbf{S} - \frac{1}{2}e\int_{S_2} \mathbf{B}\cdot \mathrm{d}\mathbf{S}\right) = \sin\left(\frac{1}{2}e\oint_{S} \mathbf{B}\cdot \mathrm{d}\mathbf{S}\right) = 0, \tag{26}$$

where by definition the closed surface $S$ contains the monopole. Result (26), Gauss's law, and $\nabla\cdot\mathbf{B} = 4\pi g\delta(\mathbf{x})$ immediately yields the Dirac condition (1) since the vanishing of the sine function in (26) requires its argument, given by

$$\frac{1}{2}e\oint_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \frac{1}{2}e\int_{V(S)} \mathrm{d}^3r\,\nabla\cdot\mathbf{B} = 2\pi eg, \tag{27}$$

to be an integer multiple of $\pi$. An identical argument shows that surfaces in the same equivalence class result in (27) vanishing since $V(S)$ does not contain the monopole. As a result, enforcing the Dirac condition removes the anomaly in the WKB approximation for the transition probability.

It is of interest to see how the general result just derived emerges for the case that a specific form for $\mathbf{A}$ is chosen and the trajectories determined in section 2 are used. In spherical coordinates $(r, \theta, \varphi)$ the vector potential $\mathbf{A}$ can be chosen to be the much studied Dirac string

$$\mathbf{A}_\pm = \frac{g(\cos\theta \pm 1)}{r\sin\theta}\hat{\boldsymbol{\varphi}}, \tag{28}$$

which yields $\mathbf{B} = \nabla\times\mathbf{A}_\pm = g\mathbf{r}/r^3$. The two forms of (28) are related by the gauge transformation $\mathbf{A}_- = \mathbf{A}_+ - \nabla\Lambda$ with $\Lambda = 2g\varphi$. The relation of the string singularity to gauge transformations is at the basis of the well-known fibre bundle formulation of the magnetic monopole first developed by Wu and Yang [4]. Fixing the $z$-axis to coincide with $-\mathbf{D}$, so that $\theta = \psi$, allows the second term in (19) to be evaluated along a classical trajectory with the result that

$$e\int_0^T \mathrm{d}t\mathbf{A}_\pm\cdot\dot{\mathbf{r}} = eg(\cos\psi \pm 1)(\varphi(T) - \varphi(0)), \tag{29}$$

where $v_\varphi = L/mr$ and (7) have been used. However, because $A_+$ and $A_-$ are related by a gauge transformation, the same result for the transition probability must occur regardless of the choice of sign in (29). This is equivalent to the demand that either equivalence class of surfaces yields the same result for (24). In the previous section it was established that there can be two classical trajectories in the limit $\gamma = 0$, one with zero winding number, for which (29) vanishes, and the other with a nonzero winding number $\varphi(T) - \varphi(0) = -2\pi k$. Using the same steps that led from (21) to (22) shows that the transition probability is gauge invariant only if

$$\cos((E_k - E_0)T + eg(\cos\psi + 1)2\pi k) - \cos((E_k - E_0)T + eg(\cos\psi - 1)2\pi k) = 0, \tag{30}$$

where $E_n$ is the energy of the winding number $n$ trajectory. The same trigonometric identity shows that (30) is satisfied if $\sin 2\pi egk = 0$ for an arbitrary integer $k$, and this immediately yields the Dirac condition (1), in agreement with the general result (26).

It was noted in section 2 that both the kinetic energy and (11) become complex if $z\tan\psi > 1$, and this places an upper bound $z_{\max} = 1/\tan(\gamma/2)$ on values of $z$ for which there is a real trajectory. Numerical analysis verifies that as $z$ approaches $z_{\max}$ the solution to (12) becomes complex valued, signalling the absence of a real trajectory. An approximate solution to (12) can be found for the case that $\gamma$ is infinitesimal. Assuming that $\psi$ is also small results in (11) giving $\varphi(T) - \varphi(0) \approx z$, so that (12) yields

$$\tan\psi \approx \sqrt{\frac{1-\cos\gamma}{\cos\gamma - \cos z}} \approx \frac{\gamma}{\sqrt{2-2\cos z}}. \tag{31}$$

Approximation (31) clearly breaks down if $z < \gamma$, again related to the appearance of complex solutions for $\psi$. Result (31) can be used in the action to demonstrate result (18). In addition, result (31) is consistent with the emergence of a free particle propagator in the $g \to 0$ limit or equivalently the $z \to 0$ limit. In order for (31) to remain valid this limit must be taken holding $\gamma \ll z$. For such a case it follows that

$$\lim_{z\to 0}\sqrt{1 - z^2\tan^2\psi} = \sqrt{1-\gamma^2} \approx 1 - \tfrac{1}{2}\gamma^2 \approx \cos\gamma \tag{32}$$

so that (10) reduces to $v_o^2 = |\,\mathbf{r}_f - \mathbf{r}_o|^2/T^2$, which is the usual free particle result.

As $z$ becomes very large the set of points connected by a real trajectory lies on and inside an infinitesimal cone containing the initial position $\mathbf{r}_o$ in its face. All other points are classically inaccessible and yield complex values for $\psi$ when (12) is solved. As a result, for large $z$ the WKB approximation can be defined only by analytic continuation to complex values of $\psi$. It is straightforward to demonstrate that there are no pure imaginary solutions to (12) for large $z$ since for pure imaginary values of $\psi = \mathrm{i}\zeta$, with $\zeta$ real, equation (12) becomes

$$\cos\gamma = \cosh^2\zeta - \sinh^2\zeta\cos\left(\frac{1}{\sinh\zeta}\sinh^{-1}(z\tanh\zeta)\right) \tag{33}$$

and the right-hand side of (33) is always greater than 1. Numerical analysis bears out this result. For example, setting $z = 10.000$ and $\gamma = 1.000$ yields an approximate solution to (12) of $\psi \approx 0.216 + 0.295\,\mathrm{i}$. As a result, the term $ET$ appearing in action (19), rewritten as

$$ET = \frac{m\left(r_f^2 + r_o^2\right)}{2T} \pm \frac{eg}{z}\sqrt{1 - z^2\tan^2\Psi}, \tag{34}$$

will be complex valued for large $z$. In that regard it is well known that $g \to -g$ under time reversal [8], so that $z$ itself is invariant. The simultaneous antiunitary operations of $T \to -T$, $g \to -g$, $\mathrm{i} \to -\mathrm{i}$, and $r_o \leftrightarrow r_f$ do not leave the WKB approximation invariant in the case that complex values for $\psi$ are used. Such a result therefore explicitly breaks time reversal invariance.

It will be of interest to analyse this unusual result in greater detail to determine an approximate form for the complex $\psi$ as well as to determine whether analytic continuation is physically relevant to the WKB approximation or whether it signals a breakdown of the WKB approximation for large $z$. However, the WKB approximation presented here can be used to determine the electron partition function since it is given by evaluating the transition element for identical initial and final states, i.e., $\gamma = 0$, so that real trajectories always exist. This analysis will be presented in future work.

## References


[1] Dirac P 1934 *Proc. R. Soc.* A **133** 60
[2] Tamm J 1931 *Zeit. F. Physik* **71** 141
Goldhaber A 1965 *Phys. Rev.* B **140** 1407
Schwinger J 1966 *Phys. Rev.* **144** 1087
[3] Fierz M 1944 *Helv. Phys. Acta* **17** 27
Peres A 1968 *Phys. Rev.* **167** 1443
Hurst A 1968 *Ann. Phys.* **50** 51

Lipkin H, Weisberg W and Peskin M 1969 *Ann. Phys.* **53** 203
Zwanziger D 1971 *Phys. Rev.* **D3** 880
Nestorov A and Aceves de la Cruz F 2002 *Phys. Lett.* A **302** 253
[4] Wu T and Yang C 1974 *J. Math. Phys.* **15** 53
Wu T and Yang C 1975 *Phys. Rev.* D **12** 3845
[5] Coleman S 1983 *The Unity of the Fundamental Interactions (Erice Lectures)* ed A Zichichi (London: Plenum)
Harvey J 1996 *Magnetic Monopoles, Duality, and Supersymmetry Preprint* hep-th/9603086
[6] Ghosh R and Pal B 2003 *Phys. Lett.* B **551** 387
Chatterjee A and Mitra P 2004 *Phys. Lett.* B **588** 217
[7] Jackiw R 1985 *Phys. Rev. Lett.* **54** 159
[8] Jackson J 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley) p 254
[9] Poincaré H 1896 *Compt. Rend.* **123** 530
[10] Lapidus I and Pietenpol J 1960 *Am. J. Phys.* **28** 17
Nadeau G 1960 *Am. J. Phys.* **28** 566
Goldhaber A 1965 *Phys. Rev.* B **140** 1407
[11] Goldstein H 1980 *Classical Mechanics* 2nd edn (Reading, MA: Addison–Wesley) p 22
[12] Schulman L 1981 *Techniques and Applications of Path Integration* (New York: Wiley)
Swanson M 1992 *Path Integrals and Quantum Processes* (New York: Academic)
[13] Morette C 1951 *Phys. Rev.* **81** 848